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Phase transitions for gauge theories on tori from the AdS/CFT correspondence

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Abstract: The vacuum of a large-N gauge field on a p-torus has a spatial stress tensor with tension along the direction of smallest periodicity and equal pressures (but p times smaller in magnitude) along the other directions, assuming an AdS/CFT correspondence and a refined form of the Horowitz-Myers positive-energy conjecture. For infinite N, the vacuum exhibits a phase transition when the lengths of the two shortest periodicities cross. A comparison is made with the Surya-Schleich-Witt phase transition at finite temperature. A zero-loop approximation is also given for large but finite N.

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Horowitz and Myers [1] have noted that one can calculate the Casimir energy density of a nonsupersymmetric Yang-Mills gauge theory on $S^1 \times R^p$, in the limit that the number N of gauge fields is made very large, by using the AdS/CFT correspondence [2–4] and the supergravity energy of what they call the AdS soliton. Here I note that if one takes the gauge theory to be defined on $R^1 \times T^p$, the product of a temporal R^1 with a spatial p-torus (the product of p S^1 circles), then in the large-N limit one gets vacuum phase transitions (e.g., in the stress tensor) when one varies the lengths of the S^1's so that the values of the two shortest lengths cross. For a thermal state of the gauge theory at temperature T, which corresponds to making the Euclidean time periodic with period $\beta = 1/T$, so that the theory is defined on a Euclidean $(p+1)$-torus, there is an additional phase transition, found previously by Surya, Schleich, and Witt [5], when β crosses the length of the shortest other period.

The AdS soliton metric in $p+2$ spacetime dimensions is [1]

$$ds^2 = \frac{r^2}{\ell^2}\left[\left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)d\tau^2 + \sum_{i=1}^{p-1}(dx^i)^2 - dt^2\right] + \left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{\ell^2}{r^2}dr^2, \qquad (1)$$

where the radial variable r ranges from r_0 to ∞, and where to avoid a conical singularity at $r = r_0$, one must make τ periodic with period $\beta_\tau = 4\pi\ell^2/(p+1)r_0$. (I have added the subscript τ to what Horowitz and Myers call simply β in order to distinguish that period, of the spatial coordinate τ, from my use of β to denote the period of the Euclidean time coordinate when I consider a thermal state.)

This soliton, an Einstein metric with cosmological constant $\Lambda = -p(p+1)/(2\ell^2)$ that is negative, represents a solution for a supergravity theory in $p+2$ dimensions in the classical limit $\ell \gg \ell_{Planck}$. By the AdS/CFT correspondence, it should be dual to a suitable state (e.g., the vacuum) of a large-N gauge theory defined on the conformal boundary of the AdS soliton metric, at $r = \infty$.

Horowitz and Myers [1] considered the case in which τ was periodic but the other spatial coordinates at constant r, the p x^i's, were not (except when they normalized the energy, which is infinite for unbounded x^i's). Then with t also unbounded, the dual gauge theory was defined on $S^1 \times R^p$ (with spatial sections $S^1 \times R^{p-1}$). However, I shall take the case in which each x^i is periodic, with period L_i for $i = 1, \cdots, p-1$. For symmetry of notation, I shall also define $x^p = \tau$ and $L_p = \beta_\tau$, so each of the p spatial coordinates for the gauge theory has period L_i, but now with $i = 1, \cdots, p$. Thus the spatial part of the manifold on which the gauge theory lives is the product of p S^1's, the p-torus T^p. I shall also take the case in which all of the fermionic fields are antiperiodic around each of the S^1's, so that in principle any of the S^1's could have length shrunk to zero at some locations in the metric of the dual supergravity theory (and hence representing a rotation by 2π at those locations, reversing the sign of fermionic fields).

Then in the case in which one is interested in the Lorentzian gauge theory (so that the Lorentzian time t has infinite range, giving an R^1 factor), the total spacetime topology on which the gauge theory lives is $R^1 \times T^p$. Up to an arbitrary (smooth, positive) conformal factor, the metric of this spacetime is what is obtained from the soliton metric (1) by

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dropping the dr^2 part, multiplying by a conformal factor l^2/r^2, and taking the limit $r \to \infty$:

$$ds^2 = -dt^2 + \sum_{i=1}^{p-1}(dx^i)^2 + d\tau^2 = -dt^2 + \sum_{i=1}^{p}(dx^i)^2. \tag{2}$$

This metric is of course flat, and the only nontrivial continuous parameters are the p lengths L_i of the S^1 factors. Since the gauge theory in this $p+1$ dimensional spacetime, dual to the supergravity theory in $p+2$ dimensions, is conformally invariant, only the $p-1$ ratios of the lengths are physically relevant for conformally invariant properties of that theory.

When the lengths L_i are all multiplied by the same positive number c, the conformally invariant gauge theory has the same physical form. Since its energy E has the dimension of inverse length, it would be multiplied by $1/c$ under this scale transformation. Thus the actual value of the energy of a CFT is not invariant under conformal transformations. However, when a representative of the conformal class of metrics for the CFT is stationary, as is the metric (2), and when any other external field coupling to the CFT is also stationary (none in our example), then in that representative metric the energy is well defined and simply scales as $1/c$ if the lengths in the metric are scaled by c under a constant conformal factor. Therefore, if the energy is multiplied by a length scale taken from the metric, the resulting product is invariant under the scaling.

In our case we can use the spatial volume to define a length scale L. If we follow Horowitz and Myers [1] to define V_{p-1} to be the volume of their $p-1$ x^i's, i.e., $V_{p-1} = L_1 L_2 \cdots L_{p-1}$, we can analogously define V_p to be the volume of our p x^i's, i.e.,

$$V_p \equiv L^p = L_1 L_2 \cdots L_{p-1} L_p = V_{p-1}\beta_\tau, \tag{3}$$

where the length scale L is thus defined to be the geometric mean of the p spatial periodicities. Then the scale-invariant quantity that reduces to the energy E when the spatial volume is scaled to unity is

$$\epsilon = EL \equiv EV_p^{1/p}, \tag{4}$$

which I shall call the scale-invariant energy.

Now by the AdS/CFT correspondence, one can equate the energy of the CFT, for some choice of scale, with the energy of the supergravity solution at the same choice of scale. Using eq. (3.16) of Horowitz and Myers [1] for the latter, one can readily calculate that the scale-invariant energy is

$$\epsilon = -C_p\left(\frac{L}{\beta_\tau}\right)^{p+1} = -C_p\left(\frac{L}{L_p}\right)^{p+1}, \tag{5}$$

$$C_p \equiv \left(\frac{4\pi}{p+1}\right)^{p+1}\frac{\ell^p}{16\pi G_{p+2}} = \frac{1}{4(p+1)G_{p+2}}\left(\frac{8\pi^2 p}{(p+1)(-\Lambda)}\right)^{\frac{p}{2}}. \tag{6}$$

This value comes from using the metric (1), in which it is the special coordinate $x^p = \tau$, with coordinate periodicity $L_p = \beta_\tau$, that has a proper length whose ratio with the proper length of each other $p-1$ x^i changes with r and goes to zero at $r = r_0$. In particular, the $p-1$ periodic x^i's for $i = 1, \cdots, p-1$ give circles whose proper lengths change in the

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same ratio as r is reduced from ∞ to r_0, and whose proper lengths never go to zero, but the proper length of the circle represented by x^p changes at a different rate with r and goes to zero at the nut [6] at $r = r_0$, a regular center of polar coordinates for the (r, x^p) two-surface.

If one filled in the conformal boundary, with metric conformal to (2), with a metric analogous to (1) but having a coordinate different from $\tau = x^p$, say x^k instead, having a nut at $r = r_0$, then one would get a supergravity solution with

$$\epsilon = \epsilon_k = -C_p \left(\frac{L}{L_k}\right)^{p+1} . \tag{7}$$

For $L_k \neq L_p$, this would correspond to a different state of the gauge theory.

Thus we see that if all of the L_i's are different, we get p different AdS solitons that can fill in the conformal boundary with representative metric (2), one for each choice of the spatial coordinate x^k that is chosen to have the nut in the interior. Each of these supergravity configurations has a different scale-invariant energy given by eq. (7).

If we chose an AdS soliton corresponding to an L_k that is not the shortest circle, then the scale-invariant energy would not be the minimum possible value for that conformal boundary. This would be a (rather trivial) counterexample to the positive-energy conjectures of Horowitz and Myers [1], assuming that one measured the energy relative to the base metric given by their eq. (4.1) and had one of their x^i's (without the nut) having a shorter period than their τ that does have the nut. (Of course, this would not be a counterexample if it is implicitly assumed that the x^i's have infinite range, as Horowitz and Myers [1] seem to do except when they assume a finite V_{p-1} in order to get a finite E.)

In any case, one could trivially rephrase the Horowitz-Myers conjectures to include the assumption that the base metric has the period of the τ coordinate shorter than the period of any of the other spatial coordinates transverse to r. If these slightly revised conjectures are true, as I shall assume here, then the lowest scale-invariant energy is the ϵ_k given by eq. (7) with L_k chosen to be the shortest S^1 in the boundary metric (2). This would then be the scale-invariant ground state energy of the gauge theory:

$$\epsilon_0 = \min_k \epsilon_k = -C_p \left(\frac{L}{\min L_k}\right)^{p+1} . \tag{8}$$

If we divide this by L, we get that the ground state energy of the gauge field in the flat spatial p-torus of edge lengths L_i is

$$E_0 = -C_p \frac{L_1 L_2 \cdots L_{k-1} L_{k+1} \cdots L_{p-1} L_p}{L_k^p}, \tag{9}$$

where L_k is the minimum of the L_i's.

From dividing this energy by the volume, and from differentiating the energy with respect to each of the edge lengths and dividing by the transverse area, one can easily get that the stress-energy tensor has only the following nonzero components, in the flat coordinate basis used in the metric (2), and with i indicating a spatial index not equal to

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the special index k that labels the shortest L_k (no sum on the repeated indices):

$$T_{00} = -\frac{C_p}{L_k^{p+1}}, \tag{10}$$

$$T_{ii} = +\frac{C_p}{L_k^{p+1}}, \tag{11}$$

$$T_{kk} = -\frac{C_p\, p}{L_k^{p+1}}. \tag{12}$$

Thus we see that the energy density is negative, there is a positive pressure of that same magnitude in each of the periodic directions except for the shortest one, and there is a negative pressure (tension) of p times that magnitude in the direction of the shortest periodic direction. As expected, the trace of the stress-energy tensor is zero.

This dependence on the spatial periodicities of the stress-energy tensor of the large-N gauge theory vacuum state in the spatial p-torus gives a vacuum phase transition whenever the periodicities are changed so that the direction of the shortest periodicity is switched. The energy density, T_{00}, is continuous but has a discontinuity in its derivative with respect to the length that either was or becomes the shortest. However, the pressures in the two directions that correspond to what was and what becomes the shortest periodicity have discontinuities, suddenly interchanging with the interchange of shortest lengths. The strong coupling apparently makes the gauge theory vacuum highly sensitive to the periodicity in the two shortest directions when they become equal.

This sudden change in the stress tensor of the strongly coupled gauge field vacuum is not similar to the smooth change in the Casimir stress tensor for a weakly coupled gauge field, so it is another feature of the difference between strong and weak coupling, besides the famous factor of 3/4 (for $p = 3$) [7].

Of course, for large but finite N, there would be no real discontinuity in the stress tensor, and no real phase transition for this system that is effectively in a finite cavity (with periodic boundary conditions for the bosons and antiperiodic boundary conditions for the fermions). However, for large N, the stress tensor would change rapidly with the two shortest periods when they are crossed, as we shall discuss later.

When one goes from the vacuum state to the thermal state at a finite temperature T for the strongly coupled gauge theory, this is equivalent to making the Euclidean time periodic with period $\beta = 1/T$, so the Euclidean metric for the gauge field would simply be the $p+1$ torus T^{p+1} with edge lengths β and the p L_i's. In this case a slight modification of the analysis above would predict that there should be a thermal phase transition when β drops below L_k, the shortest other periodicity. This is indeed what has been found [5].

For $T < 1/L_k$, [5] find one has a confinement phase, with the expectation value of the temporal Wilson loop operator being zero (in the large-N limit). The expectation value of the spatial Wilson loops along the spatial S^1's would be zero for all but the shortest S^1, which would have a nonzero expectation value for its Wilson loop. By the analysis above, using the AdS/CFT correspondence with only the purely classical supergravity solutions, one finds that the gauge field stress-energy tensor has the form given by eqs. (10)–(12),

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which thus does not change with temperature so long as it is below the transition temperature $1/L_k$. (This of course ignores the correspondence to the small effect of thermal field fluctuations about the classical supergravity solution.) Thus the strongly coupled gauge field is effectively frozen in its confined ground state, with very low specific heat (which would be slightly nonzero from the correspondence with the thermal field fluctuations about the supergravity solution).

For $T > 1/L_k$, [5] find one has a deconfinement phase, with the expectation value of the temporal Wilson loop operator being nonzero. Then the expectation value of all of the Wilson loops over the spatial S^1's would be zero (in the large-N limit). By a trivial extension of the analysis above, one finds that the gauge field stress-energy has the form

$$T_{00} = +C_p\, p\, T^{p+1}, \tag{13}$$

$$T_{ii} = T_{kk} = +C_p\, T^{p+1}. \tag{14}$$

(Here T^{p+1} denotes the temperature raised to the power that is the dimension of the spacetime in which the gauge theory is defined, not a $p+1$ torus as has been my previous use of this expression.)

This is exactly the same as the large-volume limit of a thermal gas with 3/4 (for $p = 3$) the number of degrees of freedom as the weak coupling limit of the large-N gauge field [7]. However, I emphasize that this factor of 3/4 really applies only when $\beta \ll L_k$. When β is comparable to L_k, the true thermal-Casimir stress-energy tensor of the weakly coupled gauge field would be expected to change slowly with the ratios of the periodicities, and not suddenly as its components do in eqs. (10)–(14) for large N. Thus it is not simply the factor of 3/4 that differs between the weak and strong coupling limits, but also the more detailed dependence on the periodicities.

It may be of interest to give an improved approximation for the stress-energy tensor for large but finite N when the shortest periodicities are nearly equal, which I shall do by using the zero-loop approximation for the partition function for the supergravity theory that is dual to the gauge theory.

To shorten the expressions, I shall use $n \equiv p + 1$, the dimension of the spacetime in which the conformal gauge theory lives, which in the thermal case (with periodic Euclidean time) is the flat Euclidean n-torus with orthogonal periods and with periodicity lengths L_α for $\alpha = 0, \ldots, n-1$, with $L_0 = \beta$ and with the $n-1$ other L_i's being as before. For brevity, also define the n-dimensional volume of the Euclidean spacetime to be

$$V_n \equiv V_{p+1} = \beta V_p = L_0 L_1 \cdots L_{p-1} L_p, \tag{15}$$

and use

$$C \equiv C_p \equiv C_{n-1} \equiv \left(\frac{4\pi}{n}\right)^n \frac{\ell^{n-1}}{16\pi G_{n+1}} = \frac{1}{4nG_{n+1}} \left(\frac{8\pi^2(n-1)}{-n\Lambda}\right)^{\frac{n-1}{2}}. \tag{16}$$

From [1] one can see that for $n = p+1 = 4$, $C = (\pi^2/8)N^2$, and from some examples of [2] for $n = 3$ and $n = 6$, I would conjecture that for general n, $C \propto N^{n/2}$.

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Now, as discussed above, there are n classical Euclidean supergravity solutions with this Euclidean n-torus as their conformal boundary, one for each choice of one of the n S^1's to be given a nut inside, at which the periodicity length shrinks to zero to form, along with the radial coordinate r, the center of a two-dimensional disk. If it is the coordinate γ that has the nut, then the action of that solution is

$$I_\gamma = -C\frac{V_n}{L_\gamma^n}. \tag{17}$$

Then in the zero-loop approximation, this classical solution makes a contribution to the partition function of

$$Z_\gamma = e^{-I_\gamma} = \exp\left(\frac{CV_n}{L_\gamma^n}\right). \tag{18}$$

Assuming the Horowitz-Myers conjectures [1] in the form revised above, which imply that these supergravity configurations dominate the path integral, one has that the total zero-loop partition function is

$$Z = \sum_{\gamma=0}^{n-1} Z_\gamma = \sum_{\gamma=0}^{n-1} \exp\left(\frac{CV_n}{L_\gamma^n}\right). \tag{19}$$

One can then say that each of the n classical supergravity solutions has probability

$$P_\gamma = \frac{Z_\gamma}{Z} = \exp\left(\frac{CV_n}{L_\gamma^n}\right)\Big/\sum_{\delta=0}^{n-1}\exp\left(\frac{CV_n}{L_\delta^n}\right). \tag{20}$$

Now using the toroidal symmetry of the metric and differentiating the partition function by the nontrivial parameters of the metric (the periodicity lengths L_γ) gives the following stress-energy tensor (to zero-loop approximation, which is good only for $C \gg 1$, and which ignores the correspondence to the thermal field fluctuations in the dual supergravity theory and other similar effects that would show up in a one-loop calculation for that theory):

$$T^\alpha_\beta = \sum_{\gamma=0}^{n-1} P_\gamma \frac{C}{L_\gamma^n}\left(\delta^\alpha_\beta - n\delta^\alpha_\gamma\delta^\gamma_\beta\right), \tag{21}$$

where the Einstein summation convention is not used in the last term.

Thus we see that for finite N, and hence for finite C (which goes as a power of N, with the power apparently being half the spacetime dimension n in which the gauge theory lives), there are no discontinuities in the stress-energy tensor and no true phase transitions, which agrees with what one expects on general grounds for a finite system. However, for $C \gg 1$, the stress-energy tensor changes very rapidly with the two shortest periodicities when they are very nearly equal.

For example, when the inverse temperature, $\beta \equiv 1/T \equiv L_0$, is very nearly the same as the shortest spatial periodicity, say L_k, and when all the other periodicities are significantly longer, then only

$$P_0 \approx \frac{1}{2}\left[1 + \tanh\left(\frac{CV_n}{2\beta^n} - \frac{CV_n}{2L_k^n}\right)\right] \tag{22}$$

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and

$$P_k \approx \frac{1}{2}\left[1 - \tanh\left(\frac{CV_n}{2\beta^n} - \frac{CV_n}{2L_k^n}\right)\right] \tag{23}$$

are significantly different from zero, and when they are both significantly different from zero, they change very rapidly with β and with L_k. When one integrates T_{00} over the spatial volume $V_{n-1} = V_n/\beta$, one gets, for $\beta \approx L_k$,

$$E \approx \frac{CV_{n-1}}{L_k^n}(nP_0 - 1) = -E_0(nP_0 - 1), \tag{24}$$

where the negative E_0, given by eq. (9), is the ground state energy when the inverse temperature β is taken to infinity. Then one can calculate that the specific heat is

$$\frac{dE}{dT} \approx P_0 P_k \left(\frac{nCV_{n-1}}{L_k^{n-1}}\right)^2 = P_0 P_k (-nL_k E_0)^2 = P_0 P_k n^2 C^2 \left(\frac{L}{\min L_k}\right)^{2n-2}. \tag{25}$$

Since $C \gg 1$, and since L, the geometric mean of all of the $n-1$ spatial periodicities, is larger than $\min L_k$ (and can be much larger), the specific heat can be very large when P_0 and P_k are both comparable to 1/2. Therefore, although there is not literally a phase transition for finite N (and hence finite C) and for finite V_{n-1}/L_k^{n-1}, the stress-energy tensor can change very rapidly with the temperature for large finite values of one or both of these quantities.

Thus we can conclude that in the limit of infinite N, a conformally invariant gauge theory on a flat torus (with antiperiodic boundary conditions for the fermions), dual to a supergravity theory in one higher dimension, has vacuum and thermal states that are infinitely sensitive to the two shortest periodicities of the torus when they are equal, giving a phase transition when the two shortest lengths are interchanged. This is analogous to what was previously found [5] for the thermal phase transition when the inverse temperature crosses the shortest spatial periodicity. The phase transition involves a discontinuity in the stress-energy tensor, in the components along the two shortest periodicities (either both spatial, or one being the Euclidean time periodicity for the thermal phase transition).

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